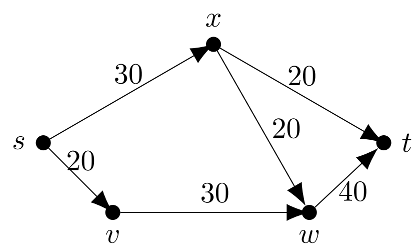
Network flows - first introduction

Original application (product of cold war)

Suppose CCCP and USA start the war. How quickly can CCCP move tank from storage s in Siberia to the target t (battle ground in Europe)? The tanks are moved on a railroad. Every link gives the capacity how many tank a day can be transported.



1: How many tanks per day can be delivered to the battleground? Is the solution unique?

Solution: 50 tanks. The solution is not unique.

Problem: A (directed) graph G, source s, sink t, capacities $u : E(G) \to \mathbb{R}^+$.
Network is (G, u, s, t).

Input: Network (G, u, s, t)
Output: s-t-flow of maximum value

s-t-**flow** f is a function $f : E(G) \to \mathbb{R}^+$ such that $f(e) \le u(e)$ for every $e \in E$.
Value of f is $\sum_{\vec{sv} \in E} f(sv) - \sum_{\vec{vs} \in E} f(vs)$ i.e. leaving − entering to s.

2: How does f look around one vertex of the network? (what axioms must f satisfy?)

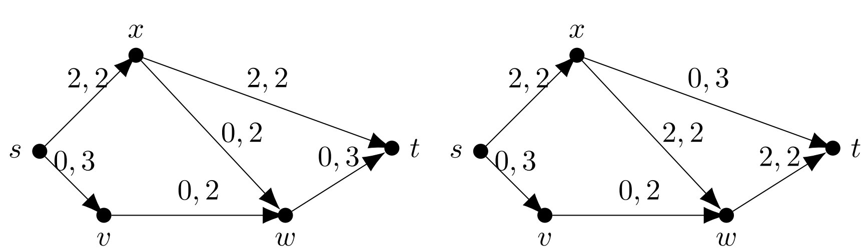
Solution: Flow conservation rule: For all $v \in V \setminus \{s, t\}$:

$$\sum_{xv \in E} f(xv) = \sum_{vx \in E} f(xv).$$

and for s, t it satisfies:

$$\sum_{sx \in E} f(sx) - \sum_{xs \in E} f(xs) = \sum_{xt \in E} f(xt) - \sum_{tx \in E} f(tx).$$

3: How do we improve these flows to be maximum? The description on each edge is the value of f, u.



Solution: Path s, v, w, t can be increased by 2. Path s, v, w, x, t can be increased by two.

4: After the improvement, how do you argue that nobody can further improve the flow?

Solution: Consider set $A = \{s, v\}$. The capacity of edges going from A is 4. Hence no flow can route more than 4 through the network.

Let $A \subset V(G)$ be such that $s \in A$ and $t \notin A$. Use $\delta^+(A)$ to denote set of edges $\overrightarrow{xy}$, where $x \in A$ and $y \notin A$ (edges leaving A). Use $\delta^-(A)$ to denote set of edges $\overrightarrow{xy}$, where $x \notin A$ and $y \in A$ (edges entering A).
Capacity of s-t-cut A is $\sum_{e\in\delta^+(A)} u(e)$.

5: Prove that for A and any flow f holds
(a) $\text{value}(f) = \sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e)$
(b) $\text{value}(f) \le \sum_{e\in\delta^+(A)} u(e)$

Solution: (a) Use conservation of flow at vertices in A.

$$\begin{aligned}\text{value}(f) &= \sum_{e\in\delta^+(s)} f(e) - \sum_{e\in\delta^-(s)} f(e)\\ &= \sum_{v\in A}\left(\sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e)\right)\\ &= \sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e)\end{aligned}$$

(b) clearly $\sum_{e\in\delta^+(A)} f(e) - \sum_{e\in\delta^-(A)} f(e) \le \sum_{e\in\delta^+(A)} u(e)$ since $f(e) \le u(e)$.

This proves the *obvious* observation that maximum flow cannot exceed capacity of minimum cut.

Notice in 3. we were improving flow be reducing the flow on $\overrightarrow{xw}$. We "sent flow in the opposite direction".

For a digraph G, define $\overleftrightarrow{G}$ by adding for every edge e also its **reverse** $\overleftarrow{e}$.

For f and u define **residual capacities** $u_f : E(\overleftrightarrow{G}) \to \mathbb{R}^+$

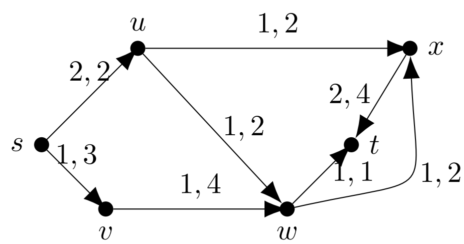
$$u_f(e) = u(e) - f(e) \qquad\qquad u_f(\overleftarrow{e}) = f(e)$$

Residual capacities ... how much extra we can send in each direction.

Residual graph G_f is obtained from $\overleftrightarrow{G}$ by removing edges $e \in E(\overleftrightarrow{G})$ with $u_f(e) = 0$.

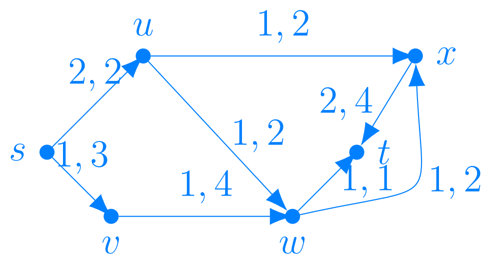
Augmenting path is an s-t path in G_f.

6: Construct the residual graph for



and find an augmenting path and increase the flow using the augmenting path.

Solution:



TODO: Too much drawing :-(

7: How do we update (i.e., **augment** the flow f) using an augmenting path in $\overleftrightarrow{G}$?

Solution: If flow is increasing on edge e by γ, then decrease $u_f(e)$ by γ and increase $u_f(\overleftarrow{e})$ by γ.

8: How to create an algorithm?

Solution: Keep finding augmenting paths and using them. The value of the value will be increasing.